GIAQUINTO ON ACQUAINTANCE WITH NUMBERS

What are finite cardinal numbers? Marcus Giaquinto writes, in this journal, that they "are properties of sets, but they might also be properties of concept extensions, collections, pluralities, nonmereological aggregates, or some other kind of collective, provided collectives of one or zero items are not excluded."[1] He also claims that they are sensible properties, and that the smallest ones are known by acquaintance. Further, he argues that this last fact can be explained using the resources of cognitive science. So there is no need for realists to adopt a supernatural rationalist account of our knowledge of small finite cardinals.

Giaquinto is arguably the most empirically informed philosopher working in this area. For this reason, it is unfortunate that authors with similar concerns —such as Jody Azzouni and Tyler Burge— have not engaged with his work. In what follows, I will do so on their behalf. I begin by comparing Giaquinto's epistemology to the Russellian one with which it invites comparison, before showing how it is subject to a version of Azzouni's "epistemic role" objection.[2] Then I argue that the source of this problem is Giaquinto's misconception that numbers, like quantities, are sensible properties. Finally, I offer a sketch of a theory of how we grasp finite cardinals on the assumption that they are not sensible – a sketch that is also consistent with findings from cognitive science.

i. ACQUAINTANCE

Giaquinto's theory of acquaintance is somewhat nonstandard. So it is worth being clear about the extent to which it accords with Russell's theory. Russell himself introduces "acquaintance" as a term of art for an immediate epistemic relation of private, subjective awareness that completely reveals the nature of the entity with which one is acquainted. I will refer to these doctrines as 'Immediacy', 'Privacy' and 'Complete Revelation'.

Beginning with Immediacy, this is the doctrine that acquaintance is not mediated by "inference or any knowledge of truths."[3] In this respect, acquaintance contrasts with the more indirect relation in which a knowing subject stands to an object, when she thinks about it via her understanding of a description that it uniquely satisfies. Certainly our grasp of the smallest finite cardinals can be contrasted with the descriptive way in which we think about larger ones, as well as about more abstract structures such as the family of structures satisfying the axioms of a ring. Because of this contrast, it is prima face worth considering whether our grasp of small finite cardinals is a kind of acquaintance.

Turning to Privacy, Russell claims that we have private awareness of among other things our own thoughts, our sense data, and our perceptually remembered experiences. Furthermore, he claims that based on our awareness of sense data, we become acquainted with their sensory properties by abstraction. For example, regarding sense data and their color properties, he says:

> by seeing many white patches, we easily learn to abstract the whiteness that they all have in common, and in learning to do this we are learning to be acquainted with whiteness.[4]

What I want to emphasize about this example is that we are not supposed to be acquainted with properties in virtue of grasping inter-subjectively accessible, shareable concepts of these things, but in virtue of abstracting them from sense data of which we are privately aware.

Russell's doctrine of Complete Revelation is that when one is acquainted with an entity one grasps it entirely, rather than in a certain limited way. This is also exemplified in his characterization of acquaintance with sensory properties:

> The particular shade of colour that I am seeing may have many things said about it... But such statements… do not make me know the colour itself any better than I did before: so far as concerns knowledge of the colour itself [by acquaintance], as opposed to knowledge of truths about it, I know the colour perfectly and completely when I see it, and no further knowledge of it itself is even theoretically possible.[5]

With that said, I now turn to Giaquinto's theory of acquaintance.

Giaquinto proposes an account of our acquaintance with sensory properties that draws on the psychological theory of category acquisition. According to this theory, we come to recognize sensory properties by the automatic and unconscious creation of categories for them, which are activated during development by repeated experience of instances. (This is supposed to explain how, for example, French infants learn to recognize the phoneme 'u' in 'tu' as distinct from 'ous' in 'vous'.) According to Giaquinto, such categories are a key ingredient of our perceptual concepts of sensory properties, and as such help facilitate acquaintance with properties. For, he claims, it suffices to be acquainted with a sensory property *F* that one has (i) perceived instances of *F* and, as a result, (ii) abstracted a perceptual concept *C* such that (iii) one can apply *C* exclusively to instances of *F* and so discriminate these from non-instances.[6]

To what extent does this view accord with Russell's doctrines about acquaintance? As regards Immediacy, Giaquinto can claim to have captured part of the grain of truth in this doctrine. This is because the process of acquiring and applying categories is supposed to be perceptual and sub-personal; further, the application of the resulting perceptual concepts is not supposed to involve conscious inference. As for Privacy, it seems open to Giaquinto to argue that acquiring and applying a perceptual concept is no part of a public practice.

Where the trouble starts is with Complete Revelation, since there is a question as to, if anything, acquaintance is supposed to reveal, and the relevant findings from cognitive science do not obviously recommend Giaquinto's view that acquaintance reveals properties. This is an application of Jody Azzouni's "epistemic role" objection, that properties play no ineliminable epistemic role in Giaquinto's theory, since conditions (i) – (iii) can be met by someone who perceives particulars and conceptualizes them appropriately. As Azzouni puts it: "We do have (conscious and nonconscious) mechanisms by which we recognize and project similarities among such objects. But descriptions of these mechanisms need nowhere posit the grasping of properties."[7] The problem is that by trying to give jointly sufficient conditions for acquaintance with properties, in a way that does not posit an unmediated relation to them, Giaquinto risks allowing properties to drop out of the epistemic picture entirely. I will return to this issue presently, as it applies to the case of finite cardinal properties.

ii. OUR SENSE OF QUANTITY

According to Giaquinto, recognition of a small finite cardinal *n* "requires that we have some sense of *n* as distinct from its neighbors."[8] But what does Giaquinto mean by "sense of *n*?" Most of us experience the phenomena of being able to estimate, visually, that there are between twenty and forty people in the room, and of being able to look at much smaller pluralities, such as three cows in a field, and see how many there are without counting. Further to these reflections, there are many disparate empirical studies purporting to establish that even prior to learning numerical concepts we are able to:

(a) Perceptually estimate the cardinal size of a given collective, and perceptually discriminate different collectives in terms of approximations of their cardinal size.

(b) Perceive the exact cardinal size of collectives of up to three or four members at a much faster rate than that required by discursive counting, an ability known as "subitizing."

The view that we share these abilities with animals has come to be widely accepted in the psychological literature, in no small part due to the work of Stanislas Dehaene, who refers to these abilities jointly as our "number sense."[9]

For example, rats can learn to press a leaver repeatedly before pressing a second lever to get a reward. Having learned to do this, they soon learn to respond with roughly the required number of presses on the first lever, before pressing the second and searching for the reward. The accuracy of their estimative capacities can then be measured by the probability of search after the wrong number of presses (the confounding quantity of duration having been controlled for). For each number of presses required by the experimenter, the mean of the distribution of the rat's responses is slightly higher than is required. Further, the standard deviation around the mean increases as a constant ratio of the mean, from which it follows that greater magnitudes must differ more than smaller ones in order for the rat to discriminate them. This accords with Weber's law,[10] which is that the discriminability of any two magnitudes is a function of their ratio: that is, the ratio of the minimum change —required to discriminate two magnitudes— to the initial magnitude is constant. Weber's law applies to representations of continuous variables such as length, area, loudness, and so conformity to it is evidence of summation of continuous magnitude rather than discrete counting. Strikingly similar results have been obtained with humans.[11]

Further experiments show that rats can also accumulate information concerning magnitude while ignoring other confounding properties of the stimuli in question. For example, they learn to press one lever in response to two flashes and another lever in response to four, before learning to press the first lever in response to two sounds and the second in response to four. Impressively, when presented with a flash synchronized with a sound, they press the lever corresponding to 2, and when presented with two flashes synchronized with two sounds they press the lever corresponding to 4. This suggests that they learn to associate different levers with different magnitudes, rather than with different perceptual modalities. In sum, it seems that the rat accumulates a continuous variable, such as a physical magnitude, which reliably correlates with the cardinal size of a given collective rather than with its other properties.

Giaquinto argues that the number sense is of use to numerate human adults, because other abilities depend on it, such as our ability to distinguish numbers during comparison tasks. This is subject to two consequences of Weber's law, namely the distance and magnitude effects. The distance effect is that the smaller the difference between two inputs the longer it takes to distinguish them. For example, it takes longer to distinguish the first pair of magnitudes than it does the second:

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The magnitude effect is that the greater the magnitude of two inputs the longer it takes to distinguish them, given a fixed difference in magnitude. Again, it takes longer to distinguish the first pair of magnitudes than it does the second:

————— ——————   —— ———

Human performance on number comparison tasks is subject to the distance and magnitude effects. As regards the former, it takes longer for adult humans to distinguish pluralities of 15 from pluralities of 10 than it does for them to distinguish pluralities of 15 from pluralities of 3. As regards the magnitude effect, it takes longer to distinguish pluralities of 15 from pluralities of 10 than it does to distinguish pluralities of 10 from pluralities of 5.[12]

An analogue accumulator model is not the only way of explaining why our estimative abilities are subject to the distance and magnitude effects, since these effects have also been simulated in a digital "thermometer" model due to Zorzi and Butterworth, which is claimed to represent the number of a given plurality discretely.[13] As is the case with the analogue accumulator, neural input is first normalized. Then a detector neuron is activated once this normalized neural input breaches a precise threshold. The detector neurons are activated incrementally and ordered by magnitude, so if the threshold of a given neuron is breached, it will activate along with all other neurons with smaller thresholds. For example, the neural representations of 4 and 6 can be pictured as follows:

4: ☒☒☒☒☐☐☐☐☐☐

6: ☒☒☒☒☒☒☐☐☐☐

Thus, according to this model, a small finite number is represented if the plurality of activated detector neurons —or neural units— is of that number. Obviously then, both ordering and cardinality are presupposed in this model. Does this render the account uninformative? Or can one assume numbers in an informative explanation of how we represent them? I will return to this point in my conclusion.

iii. ACQUAINTANCE WITH NUMBERS

Giaquinto proposes that small finite cardinals are sensible properties of collectives that we can perceive with our number sense. Further, he claims that it is sufficient to be acquainted with a finite cardinal *m* that one has (i') perceived instances of *m* with one's number sense and (ii') acquired a numerical concept of *m*, such that (iii') one can apply this concept exclusively to its instances and so discriminate these from adjacent non-instances.[14] The point is that we perceive cardinal size with our number sense, then, once we have acquired cardinal concepts during development, we use our number sense to guide our application of these concepts to very small pluralities. So we are claimed to meet a sufficient condition for being acquainted with very small finite cardinals.

Does this account of acquaintance with cardinal numbers accord with Russell's doctrines about acquaintance? In certain respects it accords with Immediacy, since the sensing of numbers and the abstraction and application of concepts is supposed to be sub-personal and perceptual, rather than conscious and discursive. Further, it accords with Privacy, since it is no part of a public practice such as counting. Again, the problem is with Complete Revelation, and in particular with the problem of what acquaintance is supposed to reveal. According to Azzouni, the problem with this sort of view is that our ability to perceptually apply numerical concepts can be fully explained without appealing to a grasp of cardinal properties. According to him:

> It should be clear that even if the various models [from cognitive science] turn out not to be quite right in their depiction of what's going on in the brain when we navigate numerical tasks, the grasping of numerical properties will nevertheless *not* be part of the empirically corrected story…
>
> One can be described as recognizing there are three objects…. One does so on the basis of the objects themselves; no immediate grasping of numerical properties is needed to explain this. Instead what's needed to explain this immediate grasping of a fact are sub-personal explanations.[15]

Further, as should be clear from the forgoing, these explanations do not mention the grasping of cardinal properties. Consider, for example, the explanation by the thermometer model of how we supposedly sense the two-ness of a given plurality. While there is an epistemic role in this theory for perceiving an object, another, and no more, it does not follow that there is a role for a further thing, the number of objects perceived.

Azzouni deploys this argument against the view that numbers are objective, and it can certainly be directed at Giaquinto in this way, since his justification for his view that numbers are objective properties of collectives is "that our explanations of people's number judgments mention the number of objects counted as a factor."[16] However, as I will explain in the next section, realists can also agree that there is no such epistemic role for numbers in Giaquinto's epistemology, for the following reason. Numbers, although objectively real, are not sensible. Quantities, in contrast, are sensible. As a result, the faculties that Giaquinto describes do not represent numbers, but only some other kind of quantity.

iv. OUR SENSE OF QUANTITY IS NOT A SENSE OF NUMBER

Beginning with the accumulator model, in my view this does not represent finite cardinal numbers (assuming there are such), because it does not represent the following constitutive properties of them: (I) discreteness, (II) potential infinity and (III) general applicability. I will now discuss each of (I) – (III) in turn.

(I) The finite cardinals are as a constitutive matter discrete. In contrast, the variable accumulated by an analog accumulator is continuous. For this reason, as Tyler Burge points out, while an analog mental representation can be correlated approximately with number, it cannot be accurate or inaccurate based upon whether or not it reflects the right discrete properties.[17] For example, it cannot accurately represent 7 as opposed to 8. But if it does not have accuracy conditions concerning discrete properties, then it cannot represent these properties at all. But then it cannot represent finite cardinals, since these are as a constitutive matter discrete.

One might try and meet this objection by appeal to the hypothesis that the accumulator accumulates a fixed unit of quantity, rather like an egg timer that is filled by pouring in cups of sand. But this hypothesis is also subject to the previous objection, since the neural analogue of one cup of sand will still be approximately one cup, and so for example will not be able to represent 7 as distinct from 7.00001.

(II) The assumption that there are potentially infinitely many sentences of English is a constraint on linguistic theorizing among cognitive scientists; further, the corresponding assumption about numbers is an equally reasonable constraint on cognitive accounts of our arithmetical capacities. But the accumulator embodies a perceptual, pre-linguistic capacity, and as such lacks the recursive or iterative capacity for potential infinity. For example, it does not have the potential to repeat the step of accumulating a fixed unit of quantity indefinitely.

(III) Because the accumulator embodies a perceptual, pre-linguistic capacity, it can only detect the sizes of concrete pluralities. But as Frege pointed out, number is not simply a property of concrete pluralities, since almost anything that can be conceptualized in terms of a suitable sortal-concept can also be numbered.[18]

Further, the thermometer model is also vulnerable to two of the objections leveled against the accumulator model, since it fails to reflect the properties described in (II) and (III): it lacks the recursive capacity to activate indefinitely many neural units, and can only reflect the sizes of concrete pluralities.

To address these problems, Giaquinto might claim that our number sense representations are integrated with numerical concepts that allow us to represent (I) – (III). As for how numerical concepts are acquired, Giaquinto offers two suggestions. The first is that: "part of the process involves mentally associating representations of initial words in the count list with initial representations supplied by the sense of numerosity [number sense]."[19] But, as we have seen, while this may be of help, it cannot be the whole story, since the number sense does not itself represent finite cardinal numbers. He also suggests that what may be required is to abstract from one's counting experience "a category representation of sets of a given size, one for each set-size from 1 to 3."[20] Based on what we are told about category acquisition, Giaquinto must mean that we automatically and unconsciously create categories for small numbers, activated by repeated experience of collectives with that many members; for example, we create a category for the number 2 activated by repeated experience of collectives with a member, another and no more. Further, since Giaquinto cites Cantor in this regard, what he has in mind must be a sub-personal analogue of Cantorian abstraction. Since, on Cantor's view, such abstraction requires abstracting away from the nature of the elements of a collective and the order in which they are given,[21] the resulting category representation will be a multitude of discrete units. Then, Giaquinto continues, these representations might "serve as representations of those cardinal numbers and get mapped onto the initial numerosity [number sense] representations."[22]

At this point one might worry that the number sense is no longer the source of acquaintance with numbers, since all the work is being done by abstraction. But perhaps the thermometer model of the number sense can help. Here the idea would be that we represent a multitude with our thermometer, and abstract from this the aforementioned Cantorian category representation, which we can then perceptually apply exclusively to instances of the relevant number, thus meeting Giaquinto's conditions for acquaintance. Further, this line of thought continues, the category representation might be an ingredient in a numerical concept that is associated with the recursive or iterative capacity for potential infinity. Thus the thermometer model can be augmented so as to reflect discrete potential infinity.

However, this speculative account will not suffice. This is because, as Frege argues, numbers cannot be represented by multitudes of units, unless the units are differentiated in some way; for if they are not differentiated, then the result of attempting to accumulate units will be one unit.[23] Visio-spatial intuition can help here, because if units were associated with a line, and accumulated one-by-one in a direction, then each accumulated unit could be individuated by its relative position on that line.[24] But this raises the worry about general applicability again. If we had to discriminate units by their positions in space in this way, then, like a measurement system, they would only apply to things that existed in space. In which case, it would still remain to explain how it is that we represent numbers, which are generally applicable, in that they can be used to number sets of abstract entities that do not do not exist in space, as well as sets of spatially located ones.[25] Giaquinto would certainly agree that numbers are not inherently spatial, and as such are generally applicable.[26] But then it remains for him to explain how we can sense them.

V. A DIAGNOSIS

To my mind, the source of the trouble here is Giaquinto's conviction that numbers are sensible properties of collectives. For example, he writes: "the answer to the question [how many sheep are there?] gives the cardinal size of the flock, and that is as much a property of the flock as its monetary value or the average age of its members."[27] But, as Frege points out, the impression that numbers are like other quantities in applying to collectives of objects is simply a confusion created by ordinary language. To see this, it suffices to remind oneself that the same collective of objects can be assigned incompatible numbers, such as when we count one flock, or ten sheep. From which it follows that numbers do not apply to collectives, since, in Frege's words, "an object to which I can ascribe different numbers with equal right is not really what has a number."[28] Rather, numbers apply to sortal-properties, which in turn apply to collectives.[29] Once this point is appreciated, it becomes clear that numbers are not sensible, since they are at least one degree of abstraction away from the sensible realm. This is why there is no epistemic role for sensing them in Giaquinto's theory.

What then of the aforementioned work in cognitive science? Has it all been in vain? Not entirely. Rather, as I will explain, some of this work is consistent with an analysis according to which the finite cardinals are properties of properties.

vi. A PROPOSAL

Saul Kripke claims that for the purposes of mathematics, one should, whenever possible, use a representation that is "structurally revelatory" – one that has a structural affinity with the subject matter it represents.[30] Consider, for example, the hereditarily finite sets ('HF'), each of which are finite and contain all possible sets that have already been formed. These can be represented using the following notation:

{Ø, {Ø}, {{Ø}}, {Ø, {Ø}}…}

{Ø, {Ø}}

Ø

Plainly it is easier to discern the content of this notation for HF, than it is to discern the content of a notation that works on the opposite principle, such that '{Ø, {Ø}, {{Ø}}, {Ø, {Ø}}…}' denotes the null set, and 'Ø' denotes the set of all possible sets of sets that have already been formed. This is because the standard notation unlike the reversed notation is structurally revelatory. But what exactly does this require?

This example shows that isomorphism is not necessary for a notation to be structurally revelatory, because the standard notation not quite isomorphic with HF. To see this, consider a set *A* under an ordering $<_a$, and another set *B* under an ordering $<_b$. There is an isomorphism between *A* and *B* just in case: (i) there is a one-to-one correspondence between members of *A* and *B* such that (ii) if x and y $\in$ *A* correspond with u and v $\in$ *B*, then $x<_a y$ iff $u<_b v$. In the present case, let *A* be the set of symbols of the standard notation (which I will now represent in bold rather than in quotes), and let *B* be HF, where *A* and *B* are under the partial orderings E_a and E_b respectively:

{Ø, {Ø, {Ø}}, {Ø, {Ø}, {{Ø}}, {Ø, {Ø}}…}}

{Ø, {Ø, {Ø}}, {Ø, {Ø}, {{Ø}},{Ø, {Ø}}…}}

There would be an isomorphism between these two structures if there were a one-to-one correspondence between their members such that, for example: (i) **Ø** and **{Ø, {Ø}}** corresponded with Ø and {Ø, {Ø}}, and (ii) **Ø**E_a**{Ø, {Ø}}** iff ØE_b{Ø, {Ø}}. However, there cannot be a one-to-one correspondence, since there is more than one order in which the same set can be represented in the standard notation. For example, recalling that by the axiom of extensionality {Ø, {Ø}} = {{Ø}, Ø}, both **{Ø, {Ø}}** and **{{Ø}, Ø}** can denote {Ø, {Ø}} and so be corresponded with it. So there is not quite an isomorphism. Nevertheless, there is a clear structural affinity between the two structures that is not present between HF and the reversed notation.

What is a structurally revelatory representation of the finite cardinals? In order to answer this, first I have to describe their structure. Developing the view that numbers are properties of properties, and assuming that the notion of equinumerosity is already understood, I define 'the number of *F*'s' as the property of being equinumerous with *F,* then define '0' as the number of an un-instantiated property *G* (such as being non-self-identical), and define '*Sm*' as the number that a property has iff it is instantiated by exactly one more individual than a property of which *m* is the number. In this way I get:

0 = the number of *G*'s

*S*0 = the number of a property that is instantiated by exactly one more individual than *G*,

*SS*0 = the number of a property that is instantiated by exactly one more individual than a property that is instantiated by exactly one more individual than *G*...

As is well known, definitions of this sort allow one to describe an initial part of a progression of so-called Frege-Russell numbers.[31] Further, while these entities are not themselves sensible or visualizable, their structural aspect is grasped via a structurally revelatory representation that we can visualize. This is because the pattern of adding exactly one more individual to the instances of a sortal-property is revealed by visualizing the pattern of accumulating units into multitudes, one-by-one in a direction (see the end of section iv). However, since what is visualized reveals only the structural aspect of the Frege-Russell numbers, what we have here is Partial rather than Complete Revelation. So this is not acquaintance with numbers.

Why does the epistemic role puzzle not make trouble for this view? I have argued that numbers cannot be represented unless the system of representation has the structure of numbers. In my view, this suffices to show that this structure has an epistemic role. Further, having these structural features seems sufficient to represent numbers, without the help of the details of the various models from cognitive science. Thus, to return to the question raised at the end of section ii, the models canvassed seem to be either insufficient, or to assume too much about what they purport to explain to be genuinely informative.[32]

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[1] Marcus Giaquinto, "Knowing Numbers," *The Journal of Philosophy* XCVIII, 1 (Jan., 2001): 5–18, at p. 7.
[2] See Jody Azzouni, *Talking About Nothing: Numbers, Hallucinations and Fictions* (New York: Oxford University Press, 2010).
[3] Bertrand Russell, *The Problems of Philosophy* (Home University Library, 1912), p.46.
[4] *ibid*, p. 101.
[5] *ibid.*
[6] I have augmented Giaquinto's account ever so slightly with one that he gives in later work, in which he also claims that one must be able to (iv) recognize instances of *F* as instances, (v) search for instances of *F* and (vi) imagine instances at will in sensory imagination. See Marcus Giaquinto, "Russell on Knowledge of Universals by Acquaintance," *Philosophy* LXXXVII, 4 (2012): 497–508.
[7] Azzouni, "Talking About Nothing," *op. cit.*, p. 33.
[8] Giaquinto, "Knowing Numbers," *op. cit.*, p. 10.
[9] Stanislas Dehaene, *The Number Sense: How the Mind Creates Mathematics* (New York: Oxford University Press, 1997).
[10] John Platt & David Johnson, "Localization of Position Within a Homogenous Behavior Chain: Effects of Error Contingencies," *Learning and Motivation* II (1971): 386–414.
[11] John Whalen, C. R. Gallistel, and Rochel Gelman. "Nonverbal counting in humans: The psychophysics of number representation." *Psychological Science* X, 2 (1999): 130–137.
[12] Dehaene, "The Number Sense," *op. cit.,* p. 61.
[13] Marco Zorzi & Brian Butterworth, "A Computational Model of Number Comparison," in Martin Hahn & Scott Stoness, eds., *Proceedings of the Twenty First Annual Conference of the Cognitive Science Society* (London: Lawrence Erlbaum Associates, 1999), pp. 772–77.
[14] Giaquinto (Jan., 2001). In this paper Giaquinto only proposes conditions (i)-(iii) as a sufficient condition for acquaintance with properties. I am not sure whether he would also claim that we meet his expanded set of conditions described in fn. 2.
[15] Azzouni, "Talking About Nothing," *op. cit.*, p. 33.
[16] Marcus Giaquinto, "What Cognitive Systems Underlie Arithmetical Abilities?" *Mind and Language* XVI, 1 (Feb., 2001): 56–68, at p. 67.
[17] See chapter 10 of Tyler Burge, *Origins of Objectivity* (New York: Oxford University Press, 2010).
[18] Gottlob Frege [1884], *The Foundations of Arithmetic: A Logico-Mathematical Enquiry into the Concept of Number* (New York: Harper & Brothers, 1953), p. 21.
[19] Giaquinto, "Knowing Numbers," *op. cit.*, p. 13.
[20] *ibid.*
[21] Georg Cantor, "Review of Frege's *Die Grundlagen*," [1895] in Peter Ebert & Marcus Rossberg, "Cantor on Frege's *Foundations of Arithmetic,*" History and Philosophy of Logic, 30, 4 (2009): 341–348.
[22] Giaquinto, "Knowing Numbers," *op. cit.*, p. 13.
[23] Frege, "The Foundations of Arithmetic," *op. cit.*, p. 50.
[24] The association of number sense representations with a visualized line is a heavily researched phenomenon. See for example Marcus Giaquinto, *Visual Thinking in Mathematics* (New York: Oxford University Press, 2007).
[25] Frege, "The Foundations of Arithmetic," *op. cit.*, p. 52.
[26] Giaquinto, "Visual Thinking in Mathematics," *op. cit.*, p. 90.
[27] Giaquinto, "Knowing Numbers," *op. cit.*, p. 7.
[28] Frege, "The Foundations of Arithmetic," *op. cit.*, p. 29.
[29] For an argument that numbers are properties of pluralities relative to a sortal property, see Nathan Salmon, "Wholes, Parts, and Numbers," *Noûs*, XXXI, 11 (1997): 1–15.
[30] Saul Kripke, *Logicism, Wittgenstein, and De Re Attitudes Towards Numbers.* Forthcoming. See also Mark Steiner, "Kripke on Logicism, Wittgenstein and De Re Beliefs about Numbers." In Alan Berger, ed., *Saul Kripke* (New York: Cambridge University Press, 2011), pp. 160–76.
[31] We can also describe a progression satisfying the axioms of arithmetic, so long as it is assumed that there exist infinitely many individuals. This observation is, of course, due to Russell. See, for example, Bertrand Russell, *Introduction to Mathematical Philosophy* (London: George Allen & Unwin, Ltd., 1919). For a detailed account of how to deduce the Frege-Russell numbers, see Peter Andrews, *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof* (Orlando: Academic Press Inc., 1986).

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